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Well-Defined Solutions of a Three-Dimensional System of Difference Equations

Merve KARA[1,*], Nouressadat TOUAFEK[2], Yasin YAZLIK[3]

[1]*Aksaray University, Ortaköy Vocational High School, 68400, Aksaray, Turkey*

[2]*Mohamed Seddik Ben Yahia University, LMAM Laboratory, Department of Mathematics, 18000, Jijel, Algeria*

[3]*Nevsehir Haci Bektas Veli University, Faculty of Science and Art, Department of Mathematics, 50300, Nevsehir, Turkey*

Highlights

- This paper focuses on a three-dimensional system of difference equations.
- The three-dimensional system of difference equations were solved.
- Asymptotic behavior of solutions were determined and periodic solutions were demonstrated.

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Abstract

We show that the three-dimensional system of difference equations

$$x_{n+1}=\frac{ax_nz_{n-1}}{z_n-\beta}+\gamma, y_{n+1}=\frac{by_nx_{n-1}}{x_n-\gamma}+\alpha,\ z_{n+1}=\frac{cz_ny_{n-1}}{y_n-\alpha}+\beta\ ,\ n\in N_0\ ,$$

where the parameters $a,b,c,\alpha,\beta,\gamma$ and the initial conditions $x_{-i},\ y_{-i},\ z_{-i},\ i\in\{0,1\}$, are non-zero real numbers, can be solved. Using the obtained formulas, we determine the asymptotic behavior of solutions and give conditions for which periodic solutions exist. Some numerical examples are given to demonstrate the theoretical results.

1. INTRODUCTION

Difference equations and their systems are related to many real life models in different branches of modern science such as biology, physics, economics, etc, in [1,2]. This is due to the fact that these models are expressed by discrete equations and this explain why difference equations and their systems have attracted the attention of many researchers in recent years in [3-6]. One of the most popular subject associated with difference equations and their system, especially the non-linear ones, is to examine their solvability and the behavior of their solutions in [7-35]. Also in these references, we can find several generalizations of solvable difference equations and systems in different ways by adding parameters, increasing the order, increasing dimensional, as in the examples given below: In an earlier paper, Elabbasy et al., in [36], dealt with, among other things, the following difference equation

$$x_{n+1}=\frac{x_nx_{n-1}}{x_n-1}+1, n\in N_0\ . \tag{1}$$

Then, in [37], Equation (1) was extended to the following two-dimensional system of difference equations

$$x_{n+1}=\frac{ax_ny_{n-1}}{y_n-\alpha}+\beta, y_{n+1}=\frac{by_nx_{n-1}}{x_n-\beta}+\alpha,\ n\in N_0, \tag{2}$$

*Corresponding author, e-mail: yyazlik@nevsehir.edu.tr

where the parameters a, b, α, β and the initial values $x_{-i}, y_{-i}, i = 0,1$, are non-zero real numbers. They obtained the forms of solutions of system (2) and investigated their behavior. Quite recently, in [38], Equation (1) and System (2) were generalized to the following difference equations system by both increasing order and taking periodic coefficients

$$x_{n+1} = \frac{a_n x_{n-k+1} y_{n-k}}{y_n - \alpha_n} + \beta_{n+1},\ y_{n+1} = \frac{b_n y_{n-k+1} x_{n-k}}{x_n - \beta_n} + \alpha_{n+1},\ n \in N_0\ , \tag{3}$$

where the sequences $(a_n)_{n\in N_0}, (b_n)_{n\in N_0}, (\alpha_n)_{n\in N_0}, (\beta_n)_{n\in N_0}$ are two periodic and the initial conditions $x_{-i}, y_{-i}, i = \overline{0,k}$, are non-zero real numbers. Also, by using obtained formulas they give the asymptotic behavior and studied the periodicity of well-defined solutions of system (3) when $a_0 = b_1$ and $a_1 = b_0$.

A natural question is if any of the corresponding three-dimensional relatives to Equation (1), Systems (2) and (3) is also solvable. Here we give a positive answer to this question, by proving that the following three-dimensional system

$$x_{n+1} = \frac{a x_n z_{n-1}}{z_n - \beta} + \gamma,\ y_{n+1} = \frac{b y_n x_{n-1}}{x_n - \gamma} + \alpha,\ z_{n+1} = \frac{c z_n y_{n-1}}{y_n - \alpha} + \beta,\ n \in N_0\ , \tag{4}$$

where the parameters $a, b, c, \alpha, \beta, \gamma$ and the initial values $x_{-i},\ y_{-i},\ z_{-i},\ i = 0,1$, are non-zero real numbers, is solvable in closed form.

The following well known result, see for example [39], will be very useful in our work.

Lemma 1. Let a, b be two real numbers and consider the first order linear difference equation

$$y_{n+1} = a y_n + b,\ n \in N_0\ . \tag{5}$$

Then the sequence $(y_n)_{n\in N_0}$, the solution of equation (5), has the form

$$y_n = \begin{cases} a^n y_0 + b\left(\frac{1-a^n}{1-a}\right), & if\ a \neq 1, \\ y_0 + bn, & if\ a = 1. \end{cases}$$

Definition 1. (Periodicity). A sequence $(x_n)_{n-k}^{\propto}$ is said to be eventually periodic with period p if there exist $n_0 \geq -k$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$, then the sequence $(x_n)_{n-k}^{\infty}$ is said to be periodic with period p.
Throughout this paper we suppose that $\prod_{j=i}^{m} A_j = 1$ and $\sum_{j=i}^{m} A_j = 0$, for all $m < i$.

2. SOLUTIONS OF SYSTEM (4)

Let $\{(x_n, y_n, z_n)\}_{n\geq -1}$ be well-defined solutions of system (4), that is the solutions such that $x_n \neq \gamma$, $y_n \neq \alpha$, $z_n \neq \beta$, for all $n \in N_0$. If one of the initial values $x_{-j}, y_{-j}, z_{-j}, j = 0,1$, is equal to zero, then the corresponding solutions of system (4) is not defined. For example, if $x_{-1} = 0$, then $y_1 = \alpha$, and so z_2 can not be calculated. Now assume that $x_{-j} \cdot y_{-j} \cdot z_{-j} \neq 0\ ,\ j = 0,1$. If one of the terms x_{n_0}, y_{n_0} and z_{n_0} for $n_0 \geq 1$, is equal to zero, then from system (4) either $x_{n_0+1} = \frac{a x_{n_0} z_{n_0-1}}{z_{n_0} - \beta} + \gamma = \gamma$ and so, it follows that, y_{n_0+2} is not defined or $y_{n_0+1} = \frac{b y_{n_0} x_{n_0-1}}{x_{n_0} - \gamma} + \alpha = \alpha$ and so, it follows that, z_{n_0+2} is not defined or $z_{n_0+1} = \frac{c z_{n_0} y_{n_0-1}}{y_{n_0} - \alpha} + \beta = \beta$ and so, it follows that, x_{n_0+2} is not defined. Thus, well-defined solutions of system (4) satisfy $x_n \cdot y_n \cdot z_n \neq 0\ ,\ n \geq -1$. By means of the change of variables

$$u_n = \frac{x_n - \gamma}{x_{n-1}}, v_n = \frac{y_n - \alpha}{y_{n-1}}, w_n = \frac{z_n - \beta}{z_{n-1}}, n \in N_0\ , \tag{6}$$

the system in (4) becomes

$$u_{n+1}=\frac{a}{w_n},\ v_{n+1}=\frac{b}{u_n}\ ,\ w_{n+1}=\frac{c}{v_n}\ ,\ n\in N_0\ . \tag{7}$$

From (7), we have three independent equations

$$u_{n+6}=u_n, v_{n+6}=v_n\ , w_{n+6}=w_n\ , n\in N_0\ , \tag{8}$$

that is the sequences $(u_n)_{n\in N_0}$, $(v_n)_{n\in N_0}$and $(w_n)_{n\in N_0}$ are periodic of period 6. So,

$$u_{6n+j}=u_j, v_{6n+j}=v_j\ , w_{6n+j}=w_j\ ,\ j=\overline{0,5},\ n\in N_0\ , \tag{9}$$

From (6) we have that

$$x_n=u_nx_{n-1}+\gamma\ , y_n=v_ny_{n-1}+\alpha\ , z_n=w_nz_{n-1}+\beta,\ n\in N_0\ . \tag{10}$$

Equations in (8) and (10) can be written in the form

$$x_{6n}=u_{6n}x_{6n-1}+\gamma=u_0x_{6n-1}+\gamma, n\in N_0\ ,$$

$$x_{6n+1}=u_{6n+1}x_{6n}+\gamma=u_1x_{6n}+\gamma, n\in N_0\ ,$$

$$x_{6n+2}=u_{6n+2}x_{6n+1}+\gamma=u_2x_{6n+1}+\gamma, n\in N_0\ ,$$

$$x_{6n+3}=u_{6n+3}x_{6n+2}+\gamma=u_3x_{6n+2}+\gamma, n\in N_0\ ,$$

$$x_{6n+4}=u_{6n+4}x_{6n+3}+\gamma=u_4x_{6n+3}+\gamma, n\in N_0\ ,$$

$$x_{6n+5}=u_{6n+5}x_{6n+4}+\gamma=u_5x_{6n+4}+\gamma, n\in N_0\ ,$$

$$x_{6n+6}=u_{6n+6}x_{6n+5}+\gamma=u_0x_{6n+5}+\gamma, n\in N_0\ ,$$

$$y_{6n}=v_{6n}y_{6n-1}+\alpha=v_0y_{6n-1}+\alpha, n\in N_0\ ,$$

$$y_{6n+1}=v_{6n+1}y_{6n}+\alpha=v_1y_{6n}+\alpha, n\in N_0\ ,$$

$$y_{6n+2}=v_{6n+2}y_{6n+1}+\alpha=v_2y_{6n+1}+\alpha, n\in N_0\ ,$$

$$y_{6n+3}=v_{6n+3}y_{6n+2}+\alpha=v_3y_{6n+2}+\alpha, n\in N_0\ ,$$

$$y_{6n+4}=v_{6n+4}y_{6n+3}+\alpha=v_4y_{6n+3}+\alpha, n\in N_0\ ,$$

$$y_{6n+5}=v_{6n+5}y_{6n+4}+\alpha=v_5y_{6n+4}+\alpha, n\in N_0\ ,$$

$$y_{6n+6}=v_{6n+6}y_{6n+5}+\alpha=v_0y_{6n+5}+\alpha, n\in N_0\ ,$$

$$z_{6n}=w_{6n}z_{6n-1}+\beta=w_0z_{6n-1}+\beta, n\in N_0\ ,$$

$$z_{6n+1}=w_{6n+1}z_{6n}+\beta=w_1z_{6n}+\beta, n\in N_0\ ,$$

$$z_{6n+2}=w_{6n+2}z_{6n+1}+\beta=w_2z_{6n+1}+\beta, n\in N_0\ ,$$

$z_{6n+3} = w_{6n+3} z_{6n+2} + \beta = w_3 z_{6n+2} + \beta, n \in N_0$,

$z_{6n+4} = w_{6n+4} z_{6n+3} + \beta = w_4 z_{6n+3} + \beta, n \in N_0$,

$z_{6n+5} = w_{6n+5} z_{6n+4} + \beta = w_5 z_{6n+4} + \beta, n \in N_0$,

$z_{6n+6} = w_{6n+6} z_{6n+5} + \beta = w_0 z_{6n+5} + \beta, n \in N_0$,

which implies that

$$x_{6(n+1)+j} = \left(\prod_{i=0}^{5} u_i\right) x_{6n+j} + \gamma \sum_{s=1}^{6} \left(\prod_{i=s+1}^{6} u_{(j+i) mod(6)}\right) , \tag{11}$$

$$y_{6(n+1)+j} = \left(\prod_{i=0}^{5} v_i\right) y_{6n+j} + \alpha \sum_{s=1}^{6} \left(\prod_{i=s+1}^{6} v_{(j+i) mod(6)}\right) , \tag{12}$$

$$z_{6(n+1)+j} = \left(\prod_{i=0}^{5} w_i\right) z_{6n+j} + \beta \sum_{s=1}^{6} \left(\prod_{i=s+1}^{6} w_{(j+i) mod(6)}\right) . \tag{13}$$

For $n \epsilon N_0$, $j = \overline{0,5}$. Let,

$$K_n^{(j)} = x_{6n+j} , L_n^{(j)} = y_{6n+j} , M_n^{(j)} = z_{6n+j}, n \in N_0,\ j = \overline{0,5}\ . \tag{14}$$

Then from (11)-(13), we get the following first order linear difference equations with constants coefficients

$$K_{n+1}^{(j)} = A K_n^{(j)} + B^{(j)}, L_{n+1}^{(j)} = C L_n^{(j)} + D^{(j)}, M_{n+1}^{(j)} = E M_n^{(j)} + F^{(j)} , n \in N_0,\ j = \overline{0,5}\ , \tag{15}$$

where

$A = \prod_{i=0}^{5} u_i \, , B^{(j)} = \gamma \sum_{s=1}^{6} \left(\prod_{i=s+1}^{6} u_{(j+i) mod(6)}\right)$,

$C = \prod_{i=0}^{5} v_i \, , D^{(j)} = \alpha \sum_{s=1}^{6} \left(\prod_{i=s+1}^{6} v_{(j+i) mod(6)}\right)$,

$E = \prod_{i=0}^{5} w_i \, , F^{(j)} = \beta \sum_{s=1}^{6} \left(\prod_{i=s+1}^{6} w_{(j+i) mod(6)}\right)$.

Solving Equations (15) using Lemma 1 and taking in account Equations (14), we obtain for $n \in N_0$ and $j = \overline{0,5}$

$$x_{6n+j} = \begin{cases} A^n x_j + B^{(j)} \left(\frac{1-A^n}{1-A}\right) & if\ A \neq 1, \\ x_j + B^{(j)} n & if\ A = 1, \end{cases} \tag{16}$$

$$y_{6n+j} = \begin{cases} C^n y_j + D^{(j)} \left(\frac{1-C^n}{1-C}\right) & if\ C \neq 1, \\ y_j + D^{(j)} n & if\ C = 1, \end{cases} \tag{17}$$

and

$$z_{6n+j} = \begin{cases} E^n z_j + F^{(j)} \left(\frac{1-E^n}{1-E}\right) & if\ E \neq 1, \\ z_j + F^{(j)} n & if\ E = 1. \end{cases} \tag{18}$$

3. PERIODICITY AND ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF SYSTEM (4)

The following results are devoted to the asymptotic behavior and the periodicity of well-defined solutions of system (4).

Theorem 1. Assume that $\{(x_n, y_n, z_n)\}_{n\geq -1}$ are well-defined solutions of system (4). Then for $j = \overline{0,5}$ the following statements hold.

a) If $(A-1)x_j + B^{(j)} \neq 0$, then we get

$$\lim_{n\to\infty}|x_{6n+j}| = \begin{cases} \left|\frac{B^{(j)}}{A-1}\right|, |A| < 1\,, \\ \infty, \quad |A| > 1\,. \end{cases}$$

Otherwise, if $(A-1)x_j + B^{(j)} = 0$ and $A \neq 1$, then $x_{6n+j} = x_j$ for all $n \in N_0$, that is the sequence $(x_n)_{n\geq -1}$ is eventually periodic with period 6 and takes the form $(x_n)_{n\geq -1} = (x_{-1}, x_0, x_1, x_2, x_3, x_4, x_5, x_0, x_1, \ldots)$.

b) If $B^{(j)} \neq 0$ and $A = 1$, then $|x_{6n+j}| \to \infty$, as $n \to \infty$. Otherwise, if $B^{(j)} = 0$ and $A = 1$, then $x_{6n+j} = x_j$, for all $n \in N_0$, that is the sequence $(x_n)_{n\geq -1}$ is periodic of period 6 and takes the form $(x_n)_{n\geq -1} = (x_{-1}, x_0, x_1, x_2, x_3, x_4, x_{-1}, x_0, x_1, \ldots)$.

c) If $(C-1)y_j + D^{(j)} \neq 0$, then we get

$$\lim_{n\to\infty}|y_{6n+j}| = \begin{cases} \left|\frac{D^{(j)}}{C-1}\right|, |C| < 1\,, \\ \infty, \quad |C| > 1\,. \end{cases}$$

Otherwise, if $(C-1)y_j + D^{(j)} = 0$ and $C \neq 1$, then $y_{6n+j} = y_j$ for all $n \in N_0$, that is the sequence $(y_n)_{n\geq -1}$ is eventually periodic with period 6 and takes the form $(y_n)_{n\geq -1} = (y_{-1}, y_0, y_1, y_2, y_3, y_4, y_5, y_0, y_1, \ldots)$.

d) If $D^{(j)} \neq 0$ and $C = 1$, then $|y_{6n+j}| \to \infty$, as $n \to \infty$. Otherwise, if $D^{(j)} = 0$ and $C = 1$, then $y_{6n+j} = y_j$, for all $n \in N_0$, that is the sequence $(y_n)_{n\geq -1}$ is periodic with period 6 and takes the form $(y_n)_{n\geq -1} = (y_{-1}, y_0, y_1, y_2, y_3, y_4, y_{-1}, y_0, y_1, \ldots)$.

e) If $(E-1)z_j + F^{(j)} \neq 0$, then we get

$$\lim_{n\to\infty}|z_{6n+j}| = \begin{cases} \left|\frac{F^{(j)}}{E-1}\right|, |E| < 1\,, \\ \infty, \quad |E| > 1\,. \end{cases}$$

Otherwise, if $(E-1)z_j + F^{(j)} = 0$ and $E \neq 1$, then $z_{6n+j} = z_j$ for all $n \epsilon N_0$, that is the sequence $(z_n)_{n\geq -1}$ is eventually periodic with period 6 and takes the form $(z_n)_{n\geq -1} = (z_{-1}, z_0, z_1, z_2, z_3, z_4, z_5, z_0, z_1, \ldots)$.

f) If $F^{(j)} \neq 0$ and $E = 1$, then $|z_{6n+j}| \to \infty$, as $n \to \infty$. Otherwise, if $F^{(j)} = 0$ and $E = 1$, then $z_{6n+j} = z_j$, for all $n \in N_0$, that is the sequence $(z_n)_{n\geq -1}$ is periodic with period 6 and takes the form $(z_n)_{n\geq -1} = (z_{-1}, z_0, z_1, z_2, z_3, z_4, z_{-1}, z_0, z_1, \ldots)$.

Proof.

We will only prove properties (a) and (b) since the other cases can be dealt with in the same manner.

a) Assume that $(A-1)x_j + B^{(j)} \neq 0$.

Clearly, if $|A| < 1$, then $|A|^n \to 0$ as $n \to \infty$. On the other hand, if $|A| > 1$, then $|A|^n \to \infty$ as $n \to \infty$. From Equation (16), we get

$$\lim_{n\to\infty}|x_{6n+j}| = \lim_{n\to\infty}\left|\frac{(A-1)x_j+B^{(j)}}{A-1}A^n + \left(\frac{B^{(j)}}{1-A}\right)\right|$$

$$= \left|\frac{(A-1)x_j+B^{(j)}}{A-1}\lim_{n\to\infty}A^n + \left(\frac{B^{(j)}}{1-A}\right)\right|$$

$$= \begin{cases} \left|\frac{B^{(j)}}{A-1}\right|, |A| < 1 \\ \infty, \quad |A| > 1 \end{cases}.$$

Now on the other hand $(A-1)x_j + B^{(j)} = 0$ and $A \neq 1$. Then we have

$$x_{6n+j} = x_j A^n + B^{(j)}\left(\frac{A^n-1}{A-1}\right)$$

$$= x_j A^n + \left(\frac{A^n-1}{A-1}\right)\left(-(A-1)x_j\right)$$

$$= x_j A^n - (A^n-1)x_j$$

$$= x_j ,$$

which completes the proof of (a).

b) Let $A = 1$. If $B^{(j)} \neq 0$, then from Equation (16) we get

$$x_{6n+j} = x_j + B^{(j)}n .$$

Letting $n \to \infty$ in above equations implies that $|x_{6n+j}| \to \infty$. On the other hand, If $B^{(j)} = 0$, then obviously, $x_{6n+j} = x_j + B^{(j)}n = x_j + 0 = x_j$, $\forall n \in N_0, j = \overline{0,5}$, which completes the proof of (b).

Corollary 1. Assume that $\{(x_n, y_n, z_n)\}_{n\geq -1}$ are well-defined solutions of system (4). Then for $j = \overline{0,5}$, the following statements hold.

a) If $A = -1$ then for all $n \in N_0$, we get

$$\begin{cases} x_{12n+j} = x_j, \\ x_{12n+6+j} = -x_j + B^{(j)}, \end{cases}$$

that is the sequence $(x_n)_{n\geq -1}$ is periodic with period 12 and takes the form

$$(x_n)_{n\geq -1} = \big(x_{-1}, x_0, x_1, x_2, x_3, x_4, -x_{-1} + B^{(5)}, -x_0 + B^{(0)}, -x_1 + B^{(1)}, -x_2 + B^{(2)}, -x_3 + B^{(3)}, -x_4 + B^{(4)}, x_{-1}, x_0, \dots\big).$$

b) If $C = -1$, then for all $n \in N_0$, we get

$$\begin{cases} y_{12n+j} = y_j, \\ y_{12n+6+j} = -y_j + D^{(j)}, \end{cases}$$

that is the sequence $(y_n)_{n\geq -1}$ is periodic with period 12 and takes the form

$$(y_n)_{n\geq -1} = \left(y_{-1}, y_0, y_1, y_2, y_3, y_4, -y_{-1} + D^{(5)}, -y_0 + D^{(0)}, -y_1 + D^{(1)}, -y_2 + D^{(2)}, -y_3 + D^{(3)}, -y_4 + D^{(4)}, y_{-1}, y_0, \dots\right).$$

c) If $E = -1$, then for all $n \in N_0$, we get

$$\begin{cases} z_{12n+j} = z_j, \\ z_{12n+6+j} = -z_j + F^{(j)}, \end{cases}$$

that is the sequence $(z_n)_{n\geq -1}$ is periodic with period 12 and takes the form

$$(z_n)_{n\geq -1} = \left(z_{-1}, z_0, z_1, z_2, z_3, z_4, -z_{-1} + F^{(5)}, -z_0 + F^{(0)}, -z_1 + F^{(1)}, -z_2 + F^{(2)}, -z_3 + F^{(3)}, -z_4 + F^{(4)}, z_{-1}, z_0, \dots\right).$$

Proof.

We will only prove property (a) since the other cases can be dealt with in the same manner.

a) When $A = -1$, then from Equation (16) we have for $n \in N_0$,

$$x_{6n+j} = (-1)^n x_j + \left(\frac{1-(-1)^n}{2}\right) B^{(j)} . \tag{19}$$

Replacing n by $2n$ and respectively by $2n + 1$ in Equation (19), we get

$$x_{12n+j} = (-1)^{2n} x_j + \left(\frac{1-(-1)^{2n}}{2}\right) B^{(j)} = x_j,$$

$$x_{12n+6+j} = (-1)^{2n+1} x_j + \left(\frac{1-(-1)^{2n+1}}{2}\right) B^{(j)} = -x_j + B^{(j)}$$

for $n \epsilon N_0$.

The following result is a direct consequence of Theorem 1 and Corollary 1.

Corollary 2. Assume that $\{(x_n, y_n, z_n)\}_{n\geq -1}$ are well-defined solutions of system (4). Then, the following statements hold.

a) If $A = C = E = -1$, then the solutions are periodic with period 12.

b) If $A = C = E = 1$ and $B^{(j)} = D^{(j)} = F^{(j)} = 0$ for $j = \overline{0,5}$, then the solutions are periodic with period 6.

c) If $|A| > 1, |C| > 1, |E| > 1$, then for $n \to \infty$ we have $|x_{6n+j}| \to \infty$, $|y_{6n+j}| \to \infty$, $|z_{6n+j}| \to \infty$ for $j = \overline{0,5}$, that is the solutions are unbounded.

4. NUMERICAL EXAMPLES

To support our theoretical results, we present numerical examples that represent the solutions of various cases of system (4).

Example 1. Consider the system (4) with the initial values $x_{-1} = 2$, $x_0 = -12$, $y_{-1} = 50$, $y_0 = 32$, $z_{-1} = 4$, $z_0 = -60$ and the parameters, $a = 1$, $b = 1$, $c = -1$, $\alpha = 3$, $\beta = 2$, $\gamma = 7$, the solutions are represented as in the following figures.

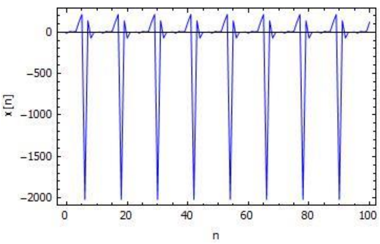

Figure 1. *Plots of* x_n

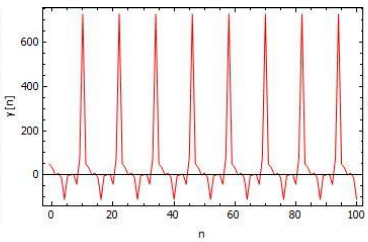

Figure 2. *Plots of* y_n

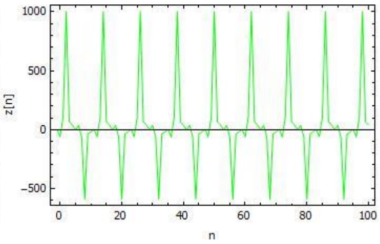

Figure 3. *Plots of* z_n

The condition (a) in Corollary 2 is satisfied from Figures (1)- (3). Therefore, the solutions of system (4) are periodic with period 12.

Example 2. Consider the system (4) with the initial values $x_{-1} = 5,\ x_0 = 4,\ y_{-1} = -2,\ y_0 = 7,\ z_{-1} = 3,\ z_0 = 9$ and the parameters, $a = b = c = 1,\ \alpha = \beta = \gamma = 0$, then the solutions are represented as follows:

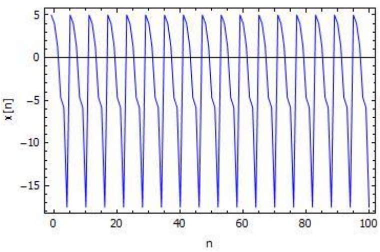

Figure 4. *Plots of* x_n

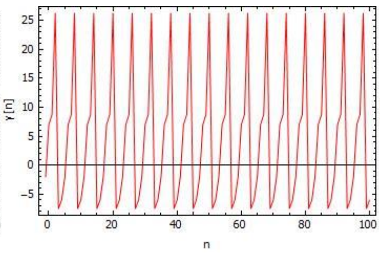

Figure 5. *Plots of* y_n

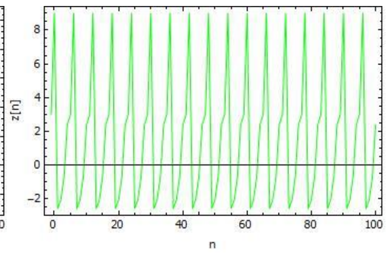

Figure 6. *Plots of* z_n

The condition (b) in Corollary 2 is satisfied from Figures (4)-(6). Therefore, the solutions of system (4) are periodic with period 6.

Example 3. Consider the system (4) with the initial values $x_{-1} = 600,\ x_0 = 700,\ y_{-1} = 80,\ y_0 = 900,\ z_{-1} = 100,\ z_0 = 110$ and the parameters, $a = 2,\ b = 4,\ c = 6,\ \alpha = 7,\ \beta = 2,\ \gamma = 5$, in this case the solutions are represented in the following figures.

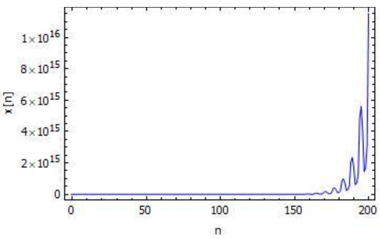

Figure 7. *Plots of* x_n

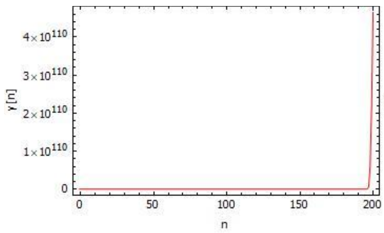

Figure 8. *Plots of* y_n

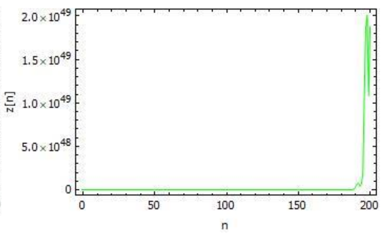

Figure 9. *Plots of* z_n

In this case the condition (c) in Corollary 2 is satisfied. From Figures (7)-(9) we can see that

$$\lim_{n\to\infty} |x_{6n+j}| \to \infty,\ \lim_{n\to\infty} |y_{6n+j}| \to \infty,\ \lim_{n\to\infty} |z_{6n+j}| \to \infty\ .$$

Example 4. Consider the system (4) with the initial values $x_{-1} = 1.3,\ x_0 = 0.2,\ y_{-1} = 2.5,\ y_0 = 0.9,\ z_{-1} = 1.8,\ z_0 = 1.1$ and the parameters, $a = 1,\ b = 2,\ c = 5,\ \alpha = 2.6,\ \beta = 1.7,\ \gamma = 0$, the solutions are represented in following figures.

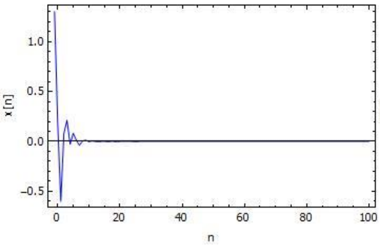

Figure 10. *Plots of* x_n

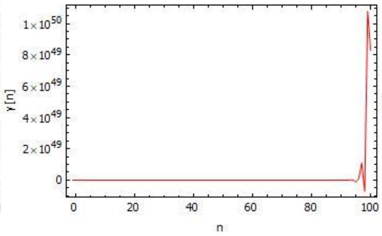

Figure 11. *Plots of* y_n

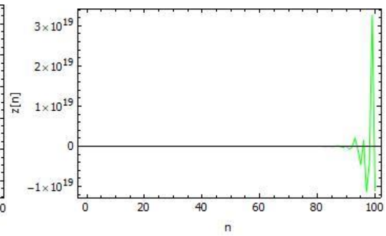

Figure 12. *Plots of* z_n

In this case the condition (a), (c), (e) in Theorem 1 is satisfied. From Figures (10)-(12) we can see that

$$\lim_{n\to\infty}|x_{6n+j}| \to 0,\ \lim_{n\to\infty}|y_{6n+j}| \to \infty,\ \ \lim_{n\to\infty}|z_{6n+j}| \to \infty.$$

Example 5. Consider the system (4) with the initial values $x_{-1} = 1.3,\ x_0 = 0.2,\ y_{-1} = 2.5,\ y_0 = 0.9,$ $z_{-1} = 1.8,\ z_0 = 1.1$ and the parameters, $a = 15,\ \ b = 4,\ \ c = 3,\ \ \alpha = 0,\ \ \beta = 5.9,\ \ \gamma = 6.3$, the following figures represent our solutions,

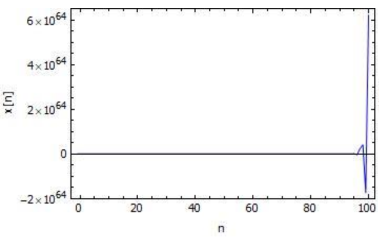

Figure 13. *Plots of* x_n

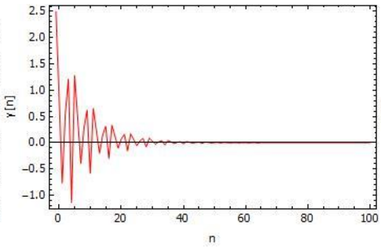

Figure 14. *Plots of* y_n

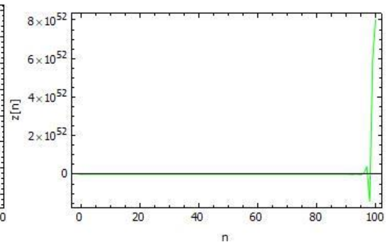

Figure 15. *Plots of* z_n

In this case the condition (a), (c), (e) in Theorem 1 is satisfied. From Figures (13)-(15) we can see that

$$\lim_{n\to\infty}|x_{6n+j}| \to \infty,\ \lim_{n\to\infty}|y_{6n+j}| \to 0,\ \ \lim_{n\to\infty}|z_{6n+j}| \to \infty.$$

Example 6. Consider the system (4) with the initial $x_{-1} = 1.3,\ x_0 = 0.2,\ y_{-1} = 2.5,\ y_0 = 0.9,\ z_{-1} = 1.8,\ z_0 = 1.1$ and the parameters, $a = 3,\ b = 16,\ c = 4,\ \alpha = 2.7,\ \beta = 0,\ \gamma = 7.9$ the solutions are represented by the following figures.

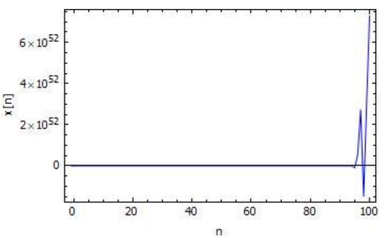

Figure 16. *Plots of* x_n

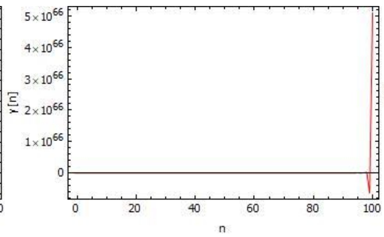

Figure 17. *Plots of* y_n

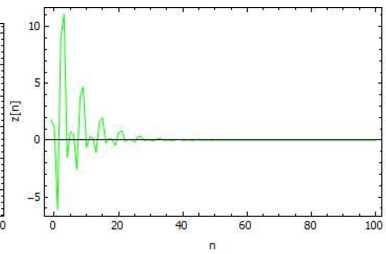

Figure 18. *Plots of* z_n

In this case the condition (a), (c), (e) in Theorem 1 is satisfied. From Figures (16)-(18) we can see that

$$\lim_{n\to\infty}|x_{6n+j}| \to \infty,\ \lim_{n\to\infty}|y_{6n+j}| \to \infty,\ \ \lim_{n\to\infty}|z_{6n+j}| \to 0.$$

5. CONCLUSION

In this study, we have consider the following three-dimensional system of difference equations
$$x_{n+1} = \frac{ax_n z_{n-1}}{z_n - \beta} + \gamma,\ \ y_{n+1} = \frac{by_n x_{n-1}}{x_n - \gamma} + \alpha,\ \ z_{n+1} = \frac{cz_n y_{n-1}}{y_n - \alpha} + \beta,\ \ n \in N_0,$$

where the parameters $a, b, c, \alpha, \beta, \gamma$ and the initial values $x_{-i}, y_{-i}, z_{-i}, i = 0,1$ are non-zero real numbers.

Firstly we have obtain the closed form of well defined solutions of the aforementioned system using suitable transformation reducing the equations of our system to linear type. Also, we have examine the behavior and the periodicity of the solutions of this system. Finally, numerical examples are provided to support our theoretical results.

It would be interesting to study the $k-$ dimensional version of the system (4).

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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