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Research Article

Computing affine equivalences and symmetries of trigonometric curves in arbitrary dimension

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Abstract

We present a new and efficient algorithm to compute affine equivalences and symmetries between two trigonometric curves in an arbitrary dimension. The algorithm benefits from the power of *invariance* and *polynomial gcd and factoring* without solving any system of equations. The algorithm is implemented in MAPLE, and extensive experimentations demonstrating the efficiency of the method are given.

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1. Introduction

Detecting affine equivalences of two varieties implies checking whether the varieties are the same in an affine setup, i.e., whether there exists a nonsingular affine transformation that maps one of the varieties to the other. It is known [3, Introduction] that detecting equivalences and symmetries of varieties in various geometries has been an attractive problem because of its contribution to the fields like Pattern Recognition [9, 18, 20], Computer Graphics [5, 8, 23], and Computer Vision [10, 21, 26]. Various approaches addressing the problem of detecting equivalences and symmetries of certain varieties can be found in some recent studies and references therein [1–4, 6, 7, 11, 13, 14] .

In this paper we investigate the problem of detecting affine equivalences between two trigonometric curves in arbitrary dimension. We refer reader to [4, 15, 16, 27] for a detailed insight about trigonometric curves. The problem is essentially solved in the paper by Alcázar and Quintero [4]. The authors provide an efficient method similar to the ones in [2, 3, 13]. However, it is essential that a method not only solves a problem completely but is also simple and computationally efficient. Thus in this paper we aim to construct a method that is simpler and more efficient than the existing ones. To do this we stick to [4] until the step where they provide their main result about detecting equivalences. A similar technique to the one given in this paper is first provided in [11], and the authors demonstrate that the new method is simpler and more efficient than the existing ones.

Our method benefits from two notions: *invariance* and *polynomial gcd and factorization*. First one is a natural consequence of the definition of geometry, i.e., the fact that geometry is characterized by its invariants. Therefore, we first determine the invariants of the affine geometry that are invariant under affine transformations, then we specialize them

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for trigonometric curves. Using these invariants, we reduce the problem to determining reparametrizations which are linear Möbius transformations. This can be done by the fact that Möbius transformation must be a factor of certain polynomial equalities (see Eqs. (4.4) and (4.5)) formed by the invariants of the curves. Finally, we determine the reparametrizations by the second main notion, i.e., polynomial gcd and factorization. To determine the equivalence itself, unlike the other approaches, we use a simple matrix multiplication without solving systems of equations.

The paper organized as follows. In Section 2 we gather some information about trigonometric curves and their affine equivalence. In Section 3, we characterize affine equivalences of trigonometric curves in arbitrary dimension using the affine invariants. In Section 4, we present our computational method and algorithm. We provide extensive tests to demonstrate the efficiency of our algorithm for the special cases, where we deal with trigonometric curves in the plane and space.

2. Preliminaries

An algebraic curve $\boldsymbol{C} \subset \mathbb{R}^n$ is called a trigonometric curve if it can be parametrized by a trigonometric parametrization [4, 16]

$$\boldsymbol{p}(t) = (p_1(t), p_2(t), \ldots, p_n(t)), \tag{2.1}$$

where

$$p_i(t) = \sum_{\ell=0}^{m_i} [a_\ell^{(i)} \cos(\ell t) + b_\ell^{(i)} \sin(\ell t)], \quad t \in [0, 2\pi],\ i = 1, \ldots, n. \tag{2.2}$$

A trigonometric parametrization $\boldsymbol{p}$ is called simple if $\boldsymbol{p}$ is injective except for finitely many parameters. Let $\boldsymbol{p}$ be a parametrization of a trigonometric curve $\boldsymbol{C}$, then a simple parametrization $\tilde{\boldsymbol{C}}$ of the same curve $\boldsymbol{C}$ is called a simplification of $\boldsymbol{p}$.

To construct our method, we need another representation of trigonometric curves called rational complex parametrization [4, 16]. This can be done by the change of parameters $z = e^{\mathbf{i}t}$. In this case the parameter space becomes the unit circle $\mathbb{S}^1$ since $z \in \mathbb{S}^1$. Given a trigonometric parametrization $\boldsymbol{p}(t)$ of a trigonometric curve $\boldsymbol{C}$. Substituting the following identities in $\boldsymbol{p}$, which can be directly obtained by $e^{\mathbf{i}t} = \cos t + \mathbf{i} \sin t$ and $\bar{z} = \frac{1}{z}$,

$$\cos(kt) = \frac{z^{2k}+1}{2z^k}, \quad \sin(kt) = \frac{z^{2k}-1}{2\mathbf{i}z^k},\ k \in \mathbb{Z},$$

we define a rational complex parametrization

$$\tilde{\boldsymbol{p}}(z) = (\tilde{p}_1(z), \tilde{p}_2(z), \ldots, \tilde{p}_n(z)), \tag{2.3}$$

whose components are $\tilde{p}_i(z) = \dfrac{P_i(z)}{z^{m_i}}$, where P_i are complex polynomials of degree $2m_i$, $i \in \{1, 2, \ldots, n\}$, and $z \in \mathbb{S}^1$. Note that $\tilde{\boldsymbol{p}}$ is a simplification of $\boldsymbol{p}$ since $\tilde{\boldsymbol{p}}$ is a simple parametrization of $\boldsymbol{C}$.

We close this section by a result already proved in [4] which characterizes the reparametrizations of trigonometric curves.

Theorem 2.1 ([4]). *Let $\tilde{\boldsymbol{p}}(z), \tilde{\boldsymbol{q}}(z)$ be two rational parametrizations of a same trigonometric curve $\boldsymbol{C}$, associated with two simple trigonometric parametrizations $\boldsymbol{p}(t), \boldsymbol{q}(t)$ of $\boldsymbol{C}$. Then there exists a Möbius transformation φ such that $\tilde{\boldsymbol{q}} = \tilde{\boldsymbol{p}} \circ \varphi$, and $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$, and $k, z \in S^1$.*

We refer the reader to [4] for a more detailed account regarding rational complex parametrisations.

3. Characterizing affine equivalences

Let $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ be two trigonometric curves in $\mathbb{R}^n$. If there exist an affine transformation $f(\boldsymbol{x}) = A\boldsymbol{x}+\boldsymbol{b}$, where $\boldsymbol{b} \in \mathbb{R}^n$ and A is a nonsingular $n \times n$ matrix, such that $f(\boldsymbol{C}_1) = \boldsymbol{C}_2$, then $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are said to be affinely equivalent. If there exists an affine transformation other than identity that leaves $\boldsymbol{C}_1$ invariant, then $\boldsymbol{C}_1$ is said to have an affine symmetry. It is well known that if A is an orthogonal matrix, then f is called an isometry. In this case, affine equivalences are reduced to Euclidean equivalences and affine symmetries are called just symmetries.

In this paper, the curves we are dealing with are the trigonometric curves $\boldsymbol{C}_1, \boldsymbol{C}_2$, none of them contained in a hyperplane of $\mathbb{R}^n$, parametrized by rational complex parametrizations $\boldsymbol{p}, \boldsymbol{q}$, respectively. Here we drop the tilde from $\tilde{\boldsymbol{p}}$ for simplicity. Using this representation, one can reduce the problem to the parameter space of the curves by the following theorem which is proved in a previous approach dealing with the same problem [4].

Theorem 3.1 ([4])**.** *Let $\boldsymbol{C}_1, \boldsymbol{C}_2$ be two trigonometric curves, none of them contained in a hyperplane, defined by rational complex parametrizations $\boldsymbol{p}(z), \boldsymbol{q}(z)$, with $z \in \mathbb{S}^1$, respectively. If the curves are affinely equivalent then there exist a nonsingular $n \times n$ matrix A, $\boldsymbol{b} \in \mathbb{R}^n$ and a Möbius transformation $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$ with $k \in \mathbb{S}^1$ such that $A\boldsymbol{p}(z) + \boldsymbol{b} = \boldsymbol{q}(\varphi(z))$.*

Now we exploit the above theorem to find an affine invariants for the trigonometric curves in an arbitrary dimension. In order to do that first we will try to get rid of the action of affine transformations on rational complex parametrizations, i.e. we will build rational functions of the parametrizations that are invariant under affinities.

Let $\boldsymbol{x}(z) = (x_1(z), x_2(z), \ldots, x_n(z))$ and $\boldsymbol{y}(z) = (y_1(z), y_2(z), \ldots, y_n(z))$ be two rational complex parametrizations and assume that $A\boldsymbol{x} + \boldsymbol{b} = \boldsymbol{y}$ with A nonsingular and $\boldsymbol{b} \in \mathbb{R}^n$. We want to determine functions F_i that are rational in the components of $\boldsymbol{x}$ and its derivatives with respect to z so that $F_i(A\boldsymbol{x}+\boldsymbol{b}) = F_i(\boldsymbol{x})$ for all nonsingular $n \times n$ matrices A and vectors $\boldsymbol{b} \in \mathbb{R}^n$. Then the functions F_i provide a simple way to check whether $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfy the relation $A\boldsymbol{x}+\boldsymbol{b} = \boldsymbol{y}$ using the fact that $F_i(A\boldsymbol{x}+\boldsymbol{b}) = F_i(\boldsymbol{y})$. The rational functions satisfying $F_i(A\boldsymbol{x} + \boldsymbol{b}) = F_i(\boldsymbol{x})$ will be called affine invariants of trigonometric curves.

We denote the determinant of the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n \in \mathbb{R}^n$ by $\|\boldsymbol{x}_1\boldsymbol{x}_2 \cdots \boldsymbol{x}_n\|$. For a nonsingular $n\times n$ matrix A, it is well-known that $\|A\boldsymbol{x}_1 A\boldsymbol{x}_2 \cdots A\boldsymbol{x}_n\| = \det(A)\|\boldsymbol{x}_1\boldsymbol{x}_2 \cdots \boldsymbol{x}_n\|$.

We also denote derivatives of a parametrization $\boldsymbol{x}$ by $\boldsymbol{x}', \boldsymbol{x}'', \ldots, \boldsymbol{x}^{(n)}, \ldots$. If $A\boldsymbol{x}+\boldsymbol{b} = \boldsymbol{y}$ then we have

$$\begin{aligned} A\boldsymbol{x}' &= \boldsymbol{y}' \\ A\boldsymbol{x}'' &= \boldsymbol{y}'' \\ &\ \vdots \\ A\boldsymbol{x}^{(n)} &= \boldsymbol{y}^{(n)} \\ &\ \vdots \end{aligned} \tag{3.1}$$

To form an $n \times n$ determinant whose columns are the vectors $\boldsymbol{x}^{(i)}$, we need n different derivative of $\boldsymbol{x}$. It does not matter which ones of them are chosen, but for simplicity we choose the ones with orders $1, 2, \ldots, n$. Then we see that

$$\|A\boldsymbol{x}' A\boldsymbol{x}'' \cdots A\boldsymbol{x}^{(n)}\| = \det(A)\|\boldsymbol{x}'\boldsymbol{x}'' \cdots \boldsymbol{x}^{(n)}\|. \tag{3.2}$$

Taking the relations in (3.1) into account and substituting them in (3.2), we get the following relation

$$\det(A)\|\boldsymbol{x}'\boldsymbol{x}'' \cdots \boldsymbol{x}^{(n)}\| = \|\boldsymbol{y}'\boldsymbol{y}'' \cdots \boldsymbol{y}^{(n)}\|. \tag{3.3}$$

Analogously, the derivatives $\boldsymbol{x}', \boldsymbol{x}'', \ldots, \boldsymbol{x}^{(n-1)}, \boldsymbol{x}^{(n+1)}$ correspond to another relation

$$\det(A)\|\boldsymbol{x}'\boldsymbol{x}''\cdots\boldsymbol{x}^{(n-1)}\boldsymbol{x}^{(n+1)}\| = \|\boldsymbol{y}'\boldsymbol{y}''\cdots\boldsymbol{y}^{(n-1)}\boldsymbol{y}^{(n+1)}\|. \tag{3.4}$$

Assuming none of the determinants in (3.3) and (3.4) vanishes and dividing them side by side, we get

$$\frac{\|\boldsymbol{x}'\boldsymbol{x}''\cdots\boldsymbol{x}^{(n-1)}\boldsymbol{x}^{(n+1)}\|}{\|\boldsymbol{x}'\boldsymbol{x}''\cdots\boldsymbol{x}^{(n)}\|} = \frac{\|\boldsymbol{y}'\boldsymbol{y}''\cdots\boldsymbol{y}^{(n-1)}\boldsymbol{y}^{(n+1)}\|}{\|\boldsymbol{y}'\boldsymbol{y}''\cdots\boldsymbol{y}^{(n)}\|}. \tag{3.5}$$

The above equality is the equality of a rational function evaluated on the parametrizations $\boldsymbol{x}$ (left hand side) and $\boldsymbol{y}$ (right hand side). We denote this function by F_n, then (3.5) can be written as $F_n(\boldsymbol{x}) = F_n(\boldsymbol{y})$. Thus the rational function F_n is an affine invariant, since F_n satisfies $F_n(A\boldsymbol{x}+\boldsymbol{b}) = F_n(\boldsymbol{x})$. Consequently we have the observation that if $A\boldsymbol{x}+\boldsymbol{b} = \boldsymbol{y}$ then $F_n(\boldsymbol{x}) = F_n(\boldsymbol{y})$.

Using the procedure above and the derivative $\boldsymbol{x}^{(n+1)}$, we can generate n other invariants in the following way. Denote the determinant function $\|\boldsymbol{x}'\boldsymbol{x}''\cdots\boldsymbol{x}^{(n)}\|$ by $\Delta(\boldsymbol{x})$. Now, to generate the rational function F_1, replace the first column $\boldsymbol{x}'$ of $\Delta(\boldsymbol{x})$ by $\boldsymbol{x}^{(n+1)}$. Denote the new determinant function by $G_1 := \|\boldsymbol{x}^{(n+1)}\boldsymbol{x}''\cdots\boldsymbol{x}^{(n)}\|$. Finally $F_1 = \dfrac{G_1}{\Delta}$. We can generalize this operation as follows. The replacement of i^{th} column of $\Delta(\boldsymbol{x})$ by $\boldsymbol{x}^{(n+1)}$ can be denoted by the function $G_i = \|\boldsymbol{x}'\cdots\boldsymbol{x}^{(i-1)}\boldsymbol{x}^{(n+1)}\boldsymbol{x}^{(i+1)}\cdots\boldsymbol{x}^{(n)}\|$ for $i \in \{2,\ldots,n-1\}$ with G_1 and G_n defined as above. Thus we have a set $\mathcal{I}$ of affine invariant functions as

$$\mathcal{I} := \left\{F_i = \frac{G_i}{\Delta} : i \in \{1,\ldots,n\}\right\}. \tag{3.6}$$

Note that if the curve $\boldsymbol{C}$ parametrized by the complex parametrization $\boldsymbol{x}(z)$ is not contained in a hyperplane then $\Delta(\boldsymbol{x})$ is not identically zero [25]. So the invariant functions in $\mathcal{I}$ are well-defined.

We see that if $\boldsymbol{x}, \boldsymbol{y}$ are two complex parametrizations and $\mathcal{I}$ is the set of affine invariants of them, then $F_i(\boldsymbol{x}) = F_i(\boldsymbol{y})$ for all $i \in \{1,\ldots,n\}$ is a necessary condition for $A\boldsymbol{x}+\boldsymbol{b} = \boldsymbol{y}$. Now we will see that $F_i(\boldsymbol{x}) = F_i(\boldsymbol{y})$ is also a sufficient condition.

Let $F_i(\boldsymbol{x}) = F_i(\boldsymbol{y})$ for all $i \in \{1,\ldots,n\}$. Let us denote the matrix function corresponding to the determinant function $\Delta(\boldsymbol{x})$ by $D(\boldsymbol{x}) := [\boldsymbol{x}'\boldsymbol{x}''\cdots\boldsymbol{x}^{(n)}]$. Since $\det(D(\boldsymbol{x})) = \delta(\boldsymbol{x})$ is not identically zero, the matrix $D(\boldsymbol{x})$ is nonsingular and hence $(D(\boldsymbol{x}))^{-1}$ is well defined. Let us consider the matrix $A := D(\boldsymbol{y})(D(\boldsymbol{x}))^{-1}$. Differentiating the matrix A, i.e. the product $D(\boldsymbol{y})(D(\boldsymbol{x}))^{-1}$, with respect to z yields

$$\begin{aligned}\frac{d(D(\boldsymbol{y})(D(\boldsymbol{x}))^{-1})}{dz} &= \frac{dD(\boldsymbol{y})}{dz}(D(\boldsymbol{x}))^{-1} + D(\boldsymbol{y})\frac{d(D(\boldsymbol{x}))^{-1}}{dz} \\ &= \frac{dD(\boldsymbol{y})}{dz}(D(\boldsymbol{x}))^{-1} - D(\boldsymbol{y})(D(\boldsymbol{x}))^{-1}\frac{dD(\boldsymbol{x})}{dz}(D(\boldsymbol{x}))^{-1} \\ &= D(\boldsymbol{y})\left((D(\boldsymbol{y}))^{-1}\frac{dD(\boldsymbol{y})}{dz} - (D(\boldsymbol{x}))^{-1}\frac{dD(\boldsymbol{x})}{dz}\right)(D(\boldsymbol{x}))^{-1}.\end{aligned} \tag{3.7}$$

For an unknown matrix U, assume that $U = (D(\boldsymbol{x}))^{-1}\dfrac{dD(\boldsymbol{x})}{dz}$. Then we get $D(\boldsymbol{x})U = \dfrac{dD(\boldsymbol{x})}{dz}$ which corresponds to n systems of equations. Each of these systems has only one solution, since $\det(D(\boldsymbol{x})) = \Delta(\boldsymbol{x})$ is not identically zero. Each solution corresponds to one column of the unknown matrix U. After solving the systems explicitly, we get

$$U = \begin{pmatrix} 0 & 0 & \cdots & 0 & F_1(\boldsymbol{x}) \\ 1 & 0 & \cdots & 0 & F_2(\boldsymbol{x}) \\ 0 & 1 & \cdots & 0 & F_3(\boldsymbol{x}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & F_n(\boldsymbol{x}) \end{pmatrix}. \tag{3.8}$$

The same process works also for an unknown matrix V so that $V = (D(\boldsymbol{y}))^{-1}\dfrac{dD(\boldsymbol{y})}{dz}$. Thus the systems $D(\boldsymbol{y})V = \dfrac{dD(\boldsymbol{y})}{dz}$ yields the solution

$$V = \begin{pmatrix} 0 & 0 & \cdots & 0 & F_1(\boldsymbol{y}) \\ 1 & 0 & \cdots & 0 & F_2(\boldsymbol{y}) \\ 0 & 1 & \cdots & 0 & F_3(\boldsymbol{y}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & F_n(\boldsymbol{y}) \end{pmatrix}. \tag{3.9}$$

By our assumption, $F_i(\boldsymbol{x}) = F_i(\boldsymbol{y})$, we get $U = V$. Substituting $U = V$ in (3.7), we conclude that $\frac{dA}{dz}$ is zero matrix, i.e. A is a constant and non-singular matrix since $D(\boldsymbol{x})$ and $D(\boldsymbol{y})$ are non-singular matrices. By $A = D(\boldsymbol{y})(D(\boldsymbol{x}))^{-1}$, it is written that $AD(\boldsymbol{x}) = D(\boldsymbol{y})$. Using the equality of the first columns, we have $A\boldsymbol{x}' = \boldsymbol{y}'$. The last differential equation yields $A\boldsymbol{x} + \boldsymbol{b} = \boldsymbol{y}$ for a constant vector $\boldsymbol{b} \in \mathbb{R}^n$.

Now let us get back to our trigonometric curves $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ defined by complex parametrizations $\boldsymbol{p}(z)$ and $\boldsymbol{q}(z)$. By the above observation, writing $\boldsymbol{x} = \boldsymbol{p}$ and $\boldsymbol{y} = \boldsymbol{q} \circ \varphi$, the following result follows.

Theorem 3.2. *Let $\boldsymbol{C}_1, \boldsymbol{C}_2$ be two trigonometric curves, none of them contained in a hyperplane, defined by rational complex parametrizations $\boldsymbol{p}(z), \boldsymbol{q}(z)$, with $z \in \mathbb{S}^1$, respectively. $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are affinely equivalent if and only if there exist a Möbius transformation $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$ with $k \in \mathbb{S}^1$ such that $F_i(\boldsymbol{p}) = F_i(\boldsymbol{q} \circ \varphi)$ for $i \in \{1, \dots, n\}$.*

The idea is to expand the expression $F_i(\boldsymbol{q} \circ \varphi)$ using the chain rule to see whether affine invariants F_i commute with φ, that is, whether $F_i(\boldsymbol{q} \circ \varphi) = F_i(\boldsymbol{q}) \circ \varphi$. However we will see that $F_i(\boldsymbol{q} \circ \varphi) \neq F_i(\boldsymbol{q}) \circ \varphi$ since k appears in all of the expressions. Thus, we will try to eliminate k from the polynomial system $F_i(\boldsymbol{p}) = F_i(\boldsymbol{q} \circ \varphi)$ to check whether there exist some other invariants which commute with φ. If they exist, they must satisfy a symmetric form like $\dfrac{A_{\boldsymbol{p}}(z)}{B_{\boldsymbol{p}}(z)} = \dfrac{A_{\boldsymbol{q}}(w)}{B_{\boldsymbol{q}}(w)}$ where $w = \varphi(z)$ and $\boldsymbol{p}, \boldsymbol{q}$ are complex parametrizations.

Let $\boldsymbol{C}_1, \boldsymbol{C}_2$ be two affinely equivalent trigonometric curves, none of them contained in a hyperplane, defined by rational complex parametrizations $\boldsymbol{p}(z), \boldsymbol{q}(z)$, with $z \in \mathbb{S}^1$, respectively. By Theorem 3.2, there exist a Möbius transformation $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$ with $k \in \mathbb{S}^1$ such that $F_i(\boldsymbol{p}) = F_i(\boldsymbol{q} \circ \varphi)$ holds. Now we want to expand $F_i(\boldsymbol{q} \circ \varphi)$. However we will compute the expansions only for the linear Möbius transformations, i.e., $\varphi(z) = kz$, since the other case can be reduced to the linear case. Let us explain how we can reduce the case where $\varphi(z) = \dfrac{k}{z}$.

If the curves are affinely equivalent, then $A\boldsymbol{p}(z) + \boldsymbol{b} = \boldsymbol{q}(\varphi(z))$, where $\varphi(z) = kz$ or $\varphi(z) = \frac{k}{z}$. Assume that we are in the case where $\varphi(z) = \frac{k}{z}$, then $A\boldsymbol{p}(z) + \boldsymbol{b} = \boldsymbol{q}(\frac{k}{z})$ holds. The latter also holds for a change of parameters $z \to \frac{1}{z}$, that is, $A\boldsymbol{p}(\frac{1}{z}) + \boldsymbol{b} = \boldsymbol{q}(kz)$. Writing $\bar{\boldsymbol{p}}(z) := \boldsymbol{p}(\frac{1}{z})$, we have $A\bar{\boldsymbol{p}}(z) + \boldsymbol{b} = \boldsymbol{q}(kz)$ which is the linear case. Thus, below, we can construct our method only for $\varphi(z) = kz$.

Let $w := \varphi(z) = kz$, we present the following lemma in which we prove a formula to expand $F_i(\boldsymbol{q}(w))$.

Lemma 3.3. *For all $i \in \{1, \ldots, n\}$, the following formula holds*

$$F_i(\boldsymbol{p})(z) = k^{n+1-i} F_i(\boldsymbol{q})(w). \tag{3.10}$$

Proof. We know that $F_i(\boldsymbol{q}(w)) = \dfrac{G_i(\boldsymbol{q}(w))}{\Delta(\boldsymbol{q}(w))}$. First we need to determine $\dfrac{d^m(\boldsymbol{q}(w))}{dz^m}$ to compute the determinants Δ and G_i. Since $w = kz$ and $w' = k$, one can easily see, using the chain rule, that

$$\frac{d^m(\boldsymbol{q}(w))}{dz^m} = k^m \boldsymbol{q}^{(m)}(w),$$

where m is a non-negative integer.

Using the above, we can easily compute the determinants $\Delta(\boldsymbol{q}(w))$ and $G_i(\boldsymbol{q}(w))$. Let us first compute $\Delta(\boldsymbol{q}(w))$. Using the definition of Δ, we have

$$\begin{aligned}\Delta(\boldsymbol{q}(w)) &= \| \frac{d(\boldsymbol{q}(w))}{dz} \frac{d^2(\boldsymbol{q}(w))}{dz^2} \cdots \frac{d^n(\boldsymbol{q}(w))}{dz^n} \| \\ &= \| k\boldsymbol{q}'(w) k^2 \boldsymbol{q}''(w) \cdots k^n \boldsymbol{q}^{(n)}(w) \| \\ &= k^{\frac{n(n+1)}{2}} \| \boldsymbol{q}'(w) \boldsymbol{q}''(w) \cdots \boldsymbol{q}^{(n)}(w) \| = k^{\frac{n(n+1)}{2}} \Delta(\boldsymbol{q})(w).\end{aligned} \tag{3.11}$$

Again, using the definition of G_i, for $i \in \{1, \ldots, n\}$, we get

$$\begin{aligned}G_i(\boldsymbol{q}(w)) &= \| \frac{d(\boldsymbol{q}(w))}{dz} \cdots \frac{d^{i-1}(\boldsymbol{q}(w))}{dz^{i-1}} \frac{d^{n+1}(\boldsymbol{q}(w))}{dz^{n+1}} \frac{d^{i+1}(\boldsymbol{q}(w))}{dz^{i+1}} \cdots \frac{d^n(\boldsymbol{q}(w))}{dz^n} \| \\ &= \| k\boldsymbol{q}'(w) \cdots k^{i-1} \boldsymbol{q}^{(i-1)}(w) k^{n+1} \boldsymbol{q}^{(n+1)}(w) k^{i+1} \boldsymbol{q}^{(i+1)}(w) \cdots k^n \boldsymbol{q}^{(n)}(w) \| \\ &= k^{\frac{n(n+1)}{2}+n+1-i} \| \boldsymbol{q}'(w) \cdots \boldsymbol{q}^{(i-1)}(w) \boldsymbol{q}^{(n+1)}(w) \boldsymbol{q}^{(i+1)}(w) \cdots \boldsymbol{q}^{(n)}(w) \| \\ &= k^{\frac{n(n+1)}{2}+n+1-i} G_i(\boldsymbol{q})(w).\end{aligned} \tag{3.12}$$

Finally, dividing Eq. (3.12) by Eq. (3.11), we have

$$F_i(\boldsymbol{q}(w)) = k^{n+1-i} F_i(\boldsymbol{q})(w).$$

□

By Lemma 3.3, the last equation of the system (3.10), i.e. the equation corresponding to $i = n$ is $F_n(\boldsymbol{q}(w)) = kF_n(\boldsymbol{q})(w)$. If we isolate k in the latter, we have $k = \dfrac{F_n(\boldsymbol{q}(w))}{F_n(\boldsymbol{q})(w)}$. Substituting this in the system (3.10), for all $i \in \{1, \ldots, n-1\}$, we get

$$F_i(\boldsymbol{q}(w)) = \frac{F_n^{n+1-i}(\boldsymbol{q}(w))}{F_n^{n+1-i}(\boldsymbol{q})(w)} F_i(\boldsymbol{q})(w), \tag{3.13}$$

which yields, for $i \in \{1, \ldots, n-1\}$,

$$\frac{F_i(\boldsymbol{q}(w))}{F_n^{n+1-i}(\boldsymbol{q}(w))} = \frac{F_i(\boldsymbol{q})(w)}{F_n^{n+1-i}(\boldsymbol{q})(w)}. \tag{3.14}$$

Let us have a closer look at the above system. If we define a rational function to be the function $K_i(\boldsymbol{q}) := \dfrac{F_i(\boldsymbol{q})}{F_n^{n+1-i}(\boldsymbol{q})}$, we rewrite Eq. (3.14) as

$$K_i(\boldsymbol{q}(w)) := K_i(\boldsymbol{q})(w), \tag{3.15}$$

where $w = kz$. The last system states that K_i which is a rational function of F_i does commute with linear Möbius transformations. Thus we have

Theorem 3.4. *Let $\boldsymbol{C}_1, \boldsymbol{C}_2$ be two trigonometric curves, none of them contained in a hyperplane, defined by rational complex parametrizations $\boldsymbol{p}(z), \boldsymbol{q}(z)$, with $z \in \mathbb{S}^1$, respectively. Also let $\bar{\boldsymbol{p}}(z) = \boldsymbol{p}(1/z)$. $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are affinely equivalent if and only if there exists a function $w := kz$ with $k \in \mathbb{S}^1$ satisfying $K_i(\boldsymbol{p}) = K_i(\boldsymbol{q})(w)$ or $K_i(\bar{\boldsymbol{p}}) = K_i(\boldsymbol{q})(w)$ for all $i \in \{1, \dots, n-1\}$ and such that $D(\boldsymbol{q}(w))(D(\boldsymbol{p}))^{-1}$ or $D(\boldsymbol{q}(w))(D(\bar{\boldsymbol{p}}))^{-1}$ is a constant matrix A and $\boldsymbol{q}(w) - A\boldsymbol{p}(z)$ or $\boldsymbol{q}(w) - A\bar{\boldsymbol{p}}(z)$ is a constant vector $\boldsymbol{b}$. Furthermore, $f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ is an affine equivalence between $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$.*

Proof. $(\Rightarrow)$: Let the curves be affinely equivalent. Then we have $A\boldsymbol{p} + \boldsymbol{b} = \boldsymbol{q}(w)$ or $A\bar{\boldsymbol{p}} + \boldsymbol{b} = \boldsymbol{q}(w)$. By Theorem 3.2 we have that $F_i(\boldsymbol{p}) = F_i(\boldsymbol{q}(w))$ or $F_i(\bar{\boldsymbol{p}}) = F_i(q(w))$ for all $i \in \{1, \dots, n\}$. The latter yields, using Eq. (3.15), that $K_i(\boldsymbol{p}) = K_i(\boldsymbol{q})(w)$ or $K_i(\bar{\boldsymbol{p}}) = K_i(\boldsymbol{q})(w)$. $(\Leftarrow)$: Let $w = kz$ be a Möbius transformation satisfying $K_i(\boldsymbol{p}) = K_i(\boldsymbol{q})(w)$ or $K_i(\bar{\boldsymbol{p}}) = K_i(\boldsymbol{q})(w)$, and making $A = D(\boldsymbol{q}(w))(D(\boldsymbol{p}))^{-1}$ or $A = D(\boldsymbol{q}(w))(D(\bar{\boldsymbol{p}}))^{-1}$ a constant matrix and $b = \boldsymbol{q}(w) - A\boldsymbol{p}$ or $b = \boldsymbol{q}(w) - A\bar{\boldsymbol{p}}$ a constant vector. By the proof of Theorem 3.2, $f(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$ is an affinity between $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$. □

4. Detecting affine equivalences

In this section we will discuss how can we detect affine equivalences simply using Theorem 3.4. Let us start by writing

$$K_i(\boldsymbol{p}) = \frac{P_i}{Q_i},\ K_i(\tilde{\boldsymbol{p}}) = \frac{\tilde{P}_i}{\tilde{Q}_i},\ K_i(\boldsymbol{q}) = \frac{R_i}{S_i} \tag{4.1}$$

where $P_i, Q_i, \tilde{P}_i, \tilde{Q}_i, R_i, S_i$ are univariate polynomials such that $gcd(P_i, Q_i) = gcd(\tilde{P}_i, \tilde{Q}_i) = gcd(R_i, S_i) = 1$. Using Theorem 3.4 and (4.1), if two trigonometric curves, $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, are affinely equivalent, then

$$\frac{P_i}{Q_i} = \frac{R_i}{S_i}, \tag{4.2}$$

or

$$\frac{\tilde{P}_i}{\tilde{Q}_i} = \frac{R_i}{S_i}. \tag{4.3}$$

Clearing the denominators in (4.2) and (4.3), we get, for $i \in \{1, \dots, n-1\}$,

$$\Phi_i(z, w) := P_i(z)S_i(w) - Q_i(z)R_i(w) = 0, \tag{4.4}$$

or

$$\tilde{\Phi}_i(z, w) := \tilde{P}_i(z)S_i(w) - \tilde{Q}_i(z)R_i(w) = 0. \tag{4.5}$$

Since we need the common factors of Φ_i or $\tilde{\Phi}_i$ for $i \in \{1, \dots, n-1\}$, it is easy to check their gcds. Thus, let us write

$$\mathcal{G}(z, w) = gcd(\Phi_1(z, w), \dots, \Phi_{n-1}(z, w)), \tag{4.6}$$

and

$$\tilde{\mathcal{G}}(z, w) = gcd(\tilde{\Phi}_1(z, w), \dots, \tilde{\Phi}_{n-1}(z, w)). \tag{4.7}$$

Finally, we are going to look for a special factor $\phi(z, w) := w - kz$ of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. We provide a simple proof for the following theorem which lets us compute the Möbius transformations just by factoring a bivariate polynomial. Thus we have,

Theorem 4.1. *Let $\boldsymbol{C}_1, \boldsymbol{C}_2$ be two trigonometric curves, none of them contained in a hyperplane, defined by rational complex parametrizations $\boldsymbol{p}(z), \boldsymbol{q}(z)$, with $z \in \mathbb{S}^1$, respectively. Also let $\mathcal{G}$ and $\tilde{\mathcal{G}}$ be as in (4.6) and (4.7). If $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are affinely equivalent, then there exists a function $\phi = w - kz$ such that ϕ divides $\mathcal{G}$ or $\tilde{\mathcal{G}}$.*

Proof. Let the curves be affinely equivalent. Then, by Theorem 3.4, there exists a Möbius transformation $\varphi(z) = kz$ satisfying $K_i(\boldsymbol{p}) = K_i(\boldsymbol{q})(\varphi)$ or $K_i(\bar{\boldsymbol{p}}) = K_i(\boldsymbol{q})(\varphi)$. Let $\phi = w - kz$ be the polynomial associated with φ. The zero set $\{(z, w) : w = kz\}$ of ϕ is contained in the set of points (z, w) satisfying $\Phi_i(z, w)$ or $\tilde{\Phi}_i(z, w)$, since φ satisfies $K_i(\boldsymbol{p}) = K_i(\boldsymbol{q})(w)$ or $K_i(\bar{\boldsymbol{p}}) = K_i(\boldsymbol{q})(w)$. Finally, since ϕ is irreducible, Bézouts theorem implies that ϕ divides Φ_i or $\tilde{\Phi}_i$, and therefore $\mathcal{G}$ or $\tilde{\mathcal{G}}$ as well. □

The above result states that in order to compute φ, we need to compute both polynomials $\mathcal{G}(z, w)$ and $\tilde{\mathcal{G}}(z, w)$, and look for the special factor ϕ by factoring them. This factorization over the complex numbers can be done by a predefined function called `AFactors` of the computer algebra system MAPLE™[22]. After determining φ, we can determine whether φ corresponds to an affine equivalence by checking whether the matrix product $A = D(\boldsymbol{q}(\varphi))(D(\boldsymbol{p}))^{-1}$ or $A = D(\boldsymbol{q}(\varphi))(D(\tilde{\boldsymbol{p}}))^{-1}$ is constant. If so, we can easily find the translation by $\boldsymbol{b} = \boldsymbol{q}(\varphi) - A\boldsymbol{p}$ or $\boldsymbol{b} = \boldsymbol{q}(\varphi) - A\tilde{\boldsymbol{p}}$. Ultimately, the following algorithm follows.

Algorithm $AffTrig$

Input: *Two trigonometric curves $\boldsymbol{C}_1, \boldsymbol{C}_2 \subset \mathbb{R}^n$, none of them contained in a hyperplane, given by the rational complex parametrizations $\boldsymbol{p}, \boldsymbol{q}$*

Output: *Either the list of Möbius transformations and affinities, or the warning: "The curves are not affinely equivalent"*

procedure $AffTrig(\boldsymbol{p}, \boldsymbol{q})$
  Compute the sets of factors $\mathcal{F}_1, \mathcal{F}_2$ of the polynomials $\mathcal{G}$ and $\tilde{\mathcal{G}}$ which are defined at (4.6) and (4.7).
  Check $\mathcal{F}_1, \mathcal{F}_2$ to find the sets $\mathcal{M}_1, \mathcal{M}_2$ of special factor ϕ.
  if $\mathcal{M}_1 = \emptyset$ and $\mathcal{M}_2 = \emptyset$ **then return** "The curves are not affinely equivalent."
  else
    Compute the set $\mathcal{S}$ of Möbius transformations corresponding to $\mathcal{M}_1 \cup \mathcal{M}_2$
    for $\varphi \in \mathcal{M}_1 \cup \mathcal{M}_2$ **do**
      Check if $D(\boldsymbol{q}(\varphi))D(\boldsymbol{p})^{-1}$ or $D(\boldsymbol{q}(\varphi))D(\tilde{\boldsymbol{p}})^{-1}$ is a constant matrix A.
      In the affirmative case, set $\boldsymbol{b} = \boldsymbol{q}(\varphi) - A\boldsymbol{p}$ or $\boldsymbol{b} = \boldsymbol{q}(\varphi) - A\tilde{\boldsymbol{p}}$ and **return** the affinity defined by A and $\boldsymbol{b}$, and the corresponding φ.
    end for
  end if
end procedure

Let us provide a detailed example in the three dimensional case, in which we can see each step of the method.

Example 4.2. Consider the trigonometric curves defined by the rational complex parametrizations

$$\boldsymbol{p}(z) = \begin{pmatrix} \dfrac{(1+2\mathbf{i})z^4 + 2z^2 + 1 - 2\mathbf{i}}{2z^2} \\ \dfrac{(-1+\mathbf{i})z^4 - 1 - \mathbf{i}}{2z^2} \\ \dfrac{(2-2\mathbf{i})z^2 + 2 + 2\mathbf{i}}{2z} \end{pmatrix}$$

$$\boldsymbol{q}(z)=\begin{pmatrix}\dfrac{-(11+7\mathbf{i})z^4+(-6+6\mathbf{i})z^3-10z^2-(6+6\mathbf{i})z-11+7\mathbf{i}}{2z^2}\\ \dfrac{-(1+2\mathbf{i})z^4+(8-8\mathbf{i})z^3-2z^2+(8+8\mathbf{i})z-1+2\mathbf{i}}{2z^2}\\ \dfrac{3z^4+4z^2+3}{2z^2}\end{pmatrix}$$

The computation of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ yields

$$\mathcal{G}(z,w)=w^2-z^2,$$
$$\tilde{\mathcal{G}}(z,w)=w^2+z^2.$$

While $\mathcal{G}$ admits the following special factors

$$\phi_1(z,w)=w-z,$$
$$\phi_2(z,w)=w+z,$$

factoring $\tilde{\mathcal{G}}$, we get the following two special factors

$$\phi_3(z,w)=w-\mathbf{i}z,$$
$$\phi_4(z,w)=w+\mathbf{i}z,$$

which yield the following four Möbius transformations

$$\varphi_1(z)=z,$$
$$\varphi_2(z)=-z,$$
$$\varphi_3(z)=\mathbf{i}z,$$
$$\varphi_4(z)=-\mathbf{i}z.$$

φ_1 corresponds to the affine transformation $f_1(\boldsymbol{x})=A_1\boldsymbol{x}+\boldsymbol{b}_1$, where

$$A_1=\begin{pmatrix}-6&5&-3\\-1&0&4\\1&-2&0\end{pmatrix},\quad \boldsymbol{b}_1=\begin{pmatrix}1\\0\\1\end{pmatrix},$$

φ_2 corresponds to the affine transformation $f_2(\boldsymbol{x})=A_2\boldsymbol{x}+\boldsymbol{b}_2$, where

$$A_2=\begin{pmatrix}-6&5&3\\-1&0&-4\\1&-2&0\end{pmatrix},\quad \boldsymbol{b}_2=\begin{pmatrix}1\\0\\1\end{pmatrix},$$

φ_3 corresponds to the affine transformation $f_3(\boldsymbol{x})=A_3\boldsymbol{x}+\boldsymbol{b}_3$, where

$$A_3=\begin{pmatrix}\frac{4}{3}&-\frac{29}{3}&-3\\-\frac{1}{3}&-\frac{4}{3}&4\\-1&2&0\end{pmatrix},\quad \boldsymbol{b}_3=\begin{pmatrix}-\frac{19}{3}\\-\frac{2}{3}\\3\end{pmatrix},$$

and φ_4 corresponds to the affine transformation $f_4(\boldsymbol{x}) = A_4\boldsymbol{x} + \boldsymbol{b}_4$, where

$$A_4 = \begin{pmatrix} \frac{4}{3} & -\frac{29}{3} & 3 \\ -\frac{1}{3} & -\frac{4}{3} & -4 \\ -1 & 2 & 0 \end{pmatrix}, \quad \boldsymbol{b}_4 = \begin{pmatrix} -\frac{19}{3} \\ -\frac{2}{3} \\ 3 \end{pmatrix}.$$

The whole computation of this example took 0.125 seconds.

5. Experimentation

In this section we provide some examples to illustrate and compare the performance of the method. In the first subsection, we consider the plane case, where we compute (affine) symmetries of well-known Lissajous curves with a rational frequency ratio. In the second subsection we consider the space curves, where we provide a comprehensive table in which we compare our algorithm with the previous one [4]. All of these examples are computed in the computer algebra system MAPLE [22], and executed in a PC with a 1.70 GHz Intel Core $i5$ processor and 6 GB RAM. MAPLE worksheets and codes are accessible in [12].

5.1. Plane case: Symmetries of Lissajous curves

A Lissajous curve is a harmonic curve given by the parametric form [17,19,24]

$$\boldsymbol{x}(t) = \begin{pmatrix} a\sin(\alpha t + \delta) \\ b\sin(\beta t) \end{pmatrix}, \tag{5.1}$$

where $a, b, \alpha, \beta, \delta$ are constants. $\frac{a}{b}$ is called the frequency ratio. If the frequency ratio is rational, then the corresponding Lissajous curve is algebraic. Hence the curve can be parametrized by

$$\boldsymbol{x}(t) = \begin{pmatrix} a\sin(mt + \delta) \\ b\sin(nt) \end{pmatrix}, \tag{5.2}$$

where m and n are integers. In this case, we can rewrite the above parametrization as

$$\boldsymbol{x}(t) = \begin{pmatrix} a_1\sin(mt) + a_2\cos(mt) \\ b\sin(nt) \end{pmatrix}, \tag{5.3}$$

where $a_1 = a\cos(\delta)$ and $a_2 = a\sin(\delta)$. The last representation is clearly a trigonometric parametrization. From there, we can represent an algebraic Lissajous curve with a rational complex parametrization, using $e^{\mathbf{i}t} = z$,

$$\boldsymbol{p}(z) = \begin{pmatrix} \dfrac{\mu z^{2m} + \bar{\mu}}{2z^m} \\ b\dfrac{z^{2n} - 1}{2\mathbf{i}z^n} \end{pmatrix}, \tag{5.4}$$

where $\mu_1 = a_2 - a_1\mathbf{i}$, and $\bar{\mu}$ is the complex conjugate of μ.

We prepare 3 sets of examples to test the method for detecting symmetries for $\{\mu = 4, b = 1\}$, $\{\mu = 4i, b = 1\}$, and $\{\mu = 3 + 4\mathbf{i}, b = 1\}$. In these cases, the classes of rational complex parametrizations are

$$\boldsymbol{p}(z) = \begin{pmatrix} 2\dfrac{z^{2m}+1}{z^m} \\ \dfrac{z^{2n}-1}{2\mathbf{i}z^n} \end{pmatrix}, \; \boldsymbol{p}(z) = \begin{pmatrix} 2i\dfrac{z^{2m}-1}{z^m} \\ \dfrac{z^{2n}-1}{2\mathbf{i}z^n} \end{pmatrix}, \; \boldsymbol{p}(z) = \begin{pmatrix} \dfrac{(3+4\mathbf{i})z^{2m}+(3-4\mathbf{i})}{2z^m} \\ \dfrac{z^{2n}-1}{2\mathbf{i}z^n} \end{pmatrix}. \tag{5.5}$$

In each of these cases, we generate three curves for $\{m=2, n=3\}$, $\{m=8, n=3\}$, and $\{m=10, n=9\}$.

The first class of examples. We illustrate the curves in Fig. 1. Table 1 corresponds to the computation times to detect symmetries of algebraic Lissajous curves in the first set of examples.

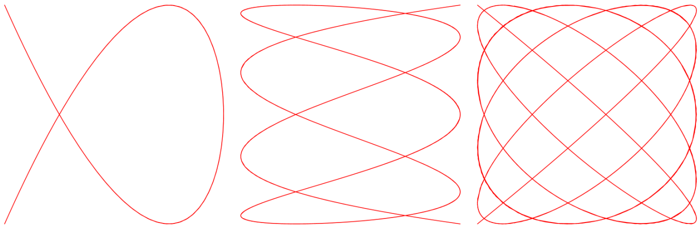

Figure 1. Graphs of Lissajous curves for the first class of examples.

m, n	t
$2, 3$	0.187
$8, 3$	0.093
$10, 9$	0.649

Table 1. Computation times in seconds for detecting symmetries in the first class of examples.

The second class of examples. We illustrate the curves in Fig. 2. Table 2 corresponds to the computation times to detect symmetries of algebraic Lissajous curves in the first set of examples.

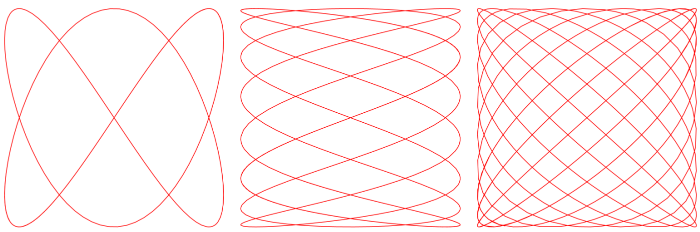

Figure 2. Graphs of Lissajous curves for the second class of examples.

m, n	t
$2, 3$	0.093
$8, 3$	0.141
$10, 9$	0.156

Table 2. Computation times in seconds for detecting symmetries in the second class of examples.

The third class of examples. We illustrate the curves in Fig. 3. Table 3 corresponds to the computation times to detect symmetries of algebraic Lissajous curves in the first set of examples.

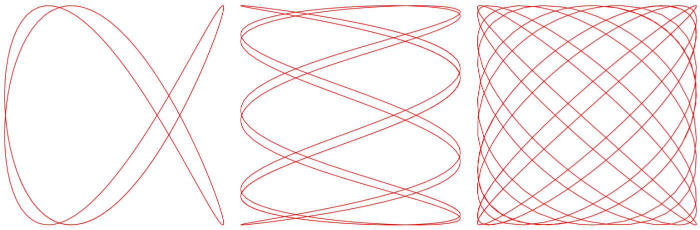

Figure 3. Graphs of Lissajous curves for the third class of examples.

m, n	t
$2, 3$	0.516
$8, 3$	1.797
$10, 9$	4.094

Table 3. Computation times in seconds for detecting symmetries in the third class of examples.

In each cases, our algorithm can determine the symmetries in at most 4.094 seconds. While, in all examples in the first and second classes, the algorithm returns 4 symmetries, it returns 2 symmetries in the examples of the third class.

5.2. Space case

In this subsection, we aim to test various space curves that are randomly generated and to compare our algorithm with the one given in [4]. In order to do that, we constructed a random trigonometric example generator in MAPLE, using chebyshev polynomials, and implemented the algorithm given in [4] since the authors provide no implementation and timings.

We first generate a random rational complex parametrization corresponding to a random trigonometric parametrization of a given degree. The coefficients of these parametrizations are chosen randomly between -16 and 16. Once we generate the parametrization, we apply the following affine transformation to the randomly generated parametrization to get another parametrization:

$$f(\boldsymbol{x}) = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Finally we compose the latter with the Möbius transformation $\varphi(z) = \mathbf{i}z$. Then we have two randomly generated trigonometric curves, given in rational complex form, which are affinely equivalent. Table 4 presents the computation times consumed to compute affine equivalences between the randomly generated curves, where t_1 and t_2 denote computation time for our algorithm and the algorithm in [4], respectively. One can see that our algorithm works better than the existing one.

degree	t_1	t_2
4	0.285	0.375
6	1.219	1.969
8	4.625	17.812
10	7.328	88.188
12	10.937	133.969

Table 4. Computation times in seconds for detecting affine equivalences of randomly generated trigonometric curves in $3D$ case

6. Conclusion

We have presented an algorithm, improving the algorithm in [4], computing affine equivalences between two trigonometric curves in an arbitrary dimension. Unlike existing algorithms, our method uses polynomial factoring instead polynomial system solving. However, similar to existing algorithms, our algorithm needs rational complex parametrizations as input. We have implemented our algorithm and the algorithm in [4] in `Maple` to present its performance, and to compare the algorithms mentioned in this paper. We provide evidences to show that our algorithm improves the previous results provided for the same problem. Due to the method we constructed here, experimentation section supports that our algorithm is simpler and works faster than the existing algorithms.

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